TADsens: Transient Adjoint DAE Sensitivities

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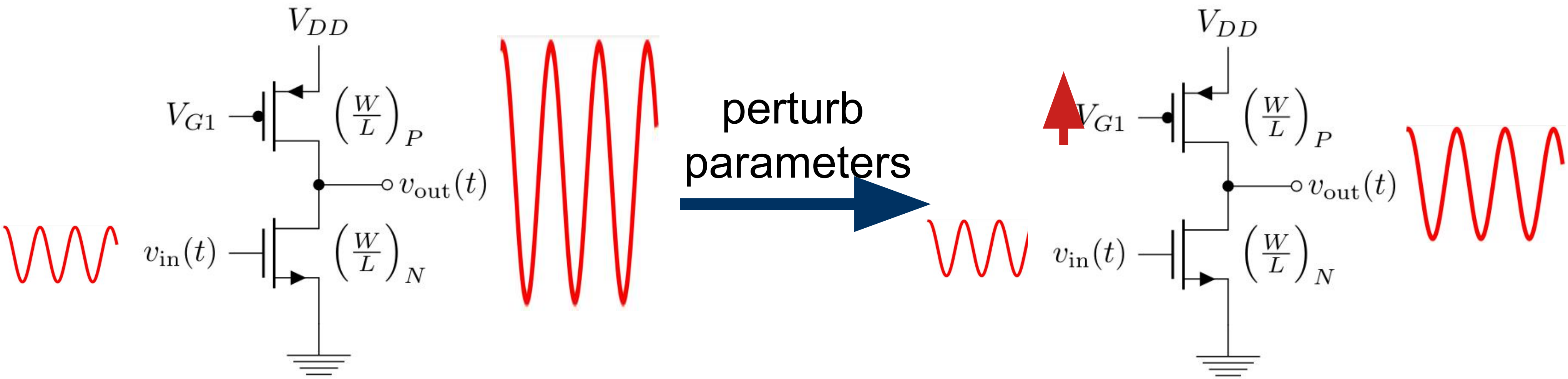
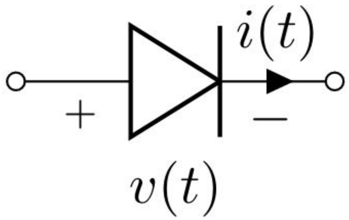
Importance of Transient Sensitivities

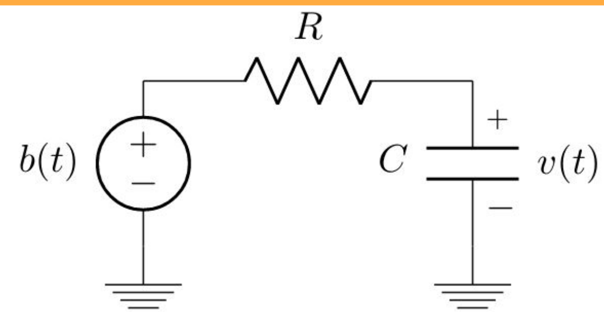
- **Context**: transient circuit simulation
- **Circuit parameters**: MOSFET dimensions, resistors, capacitors, supply voltages, *etc.*
- **Sensitivities = change in circuit output for a given change in parameter values**



Circuit Representations

Device + network equations in terms of voltages, currents

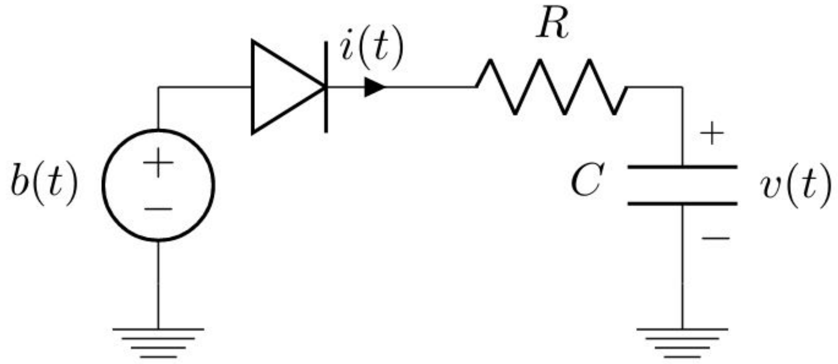




$$i(t) = I_s(e^{-v(t)/V_T} - 1)$$

$$\frac{dv}{dt} = -\frac{v(t) - u(t)}{RC}$$

DAE: system of differential and algebraic equations

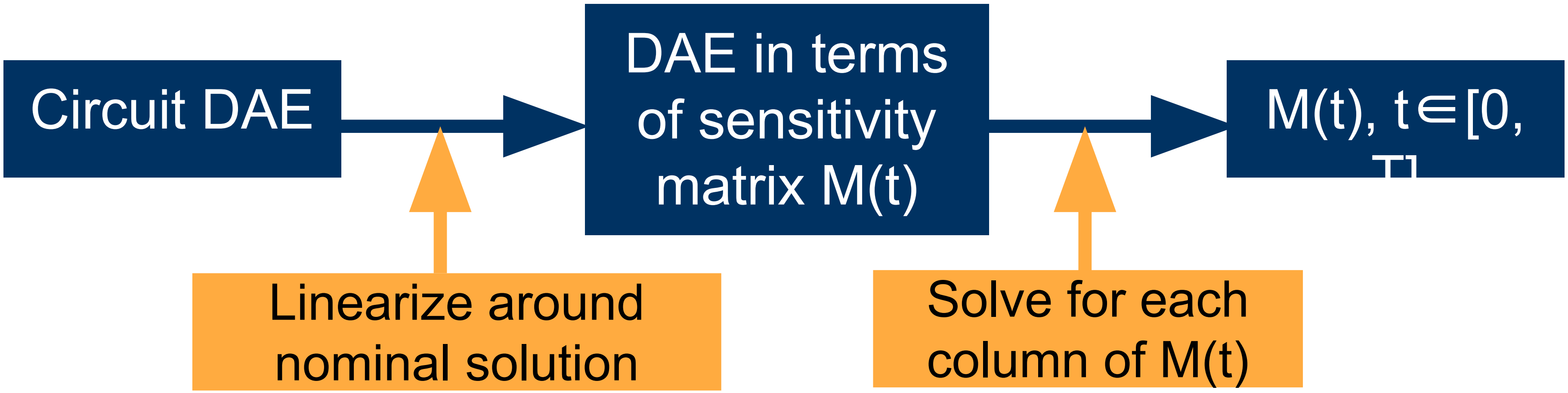


algebraic

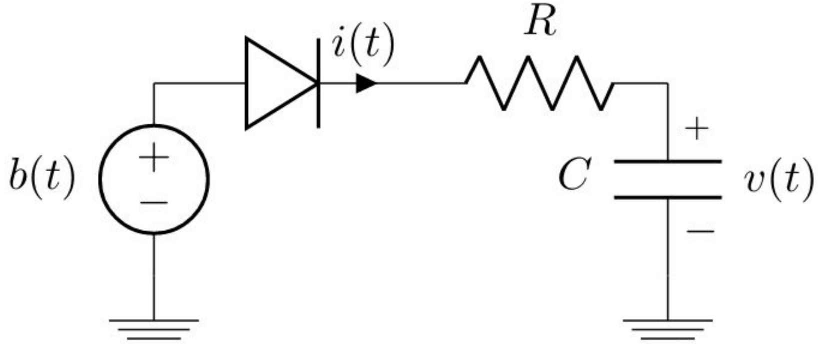
$$\begin{cases} i(t) = I_S\left(\exp\left(\frac{u(t)-i(t)R-v(t)}{V_T}\right) - 1\right) \\ \frac{dv}{dt} = \frac{i(t)}{C} \end{cases}$$

differential

Direct Sensitivity Computation

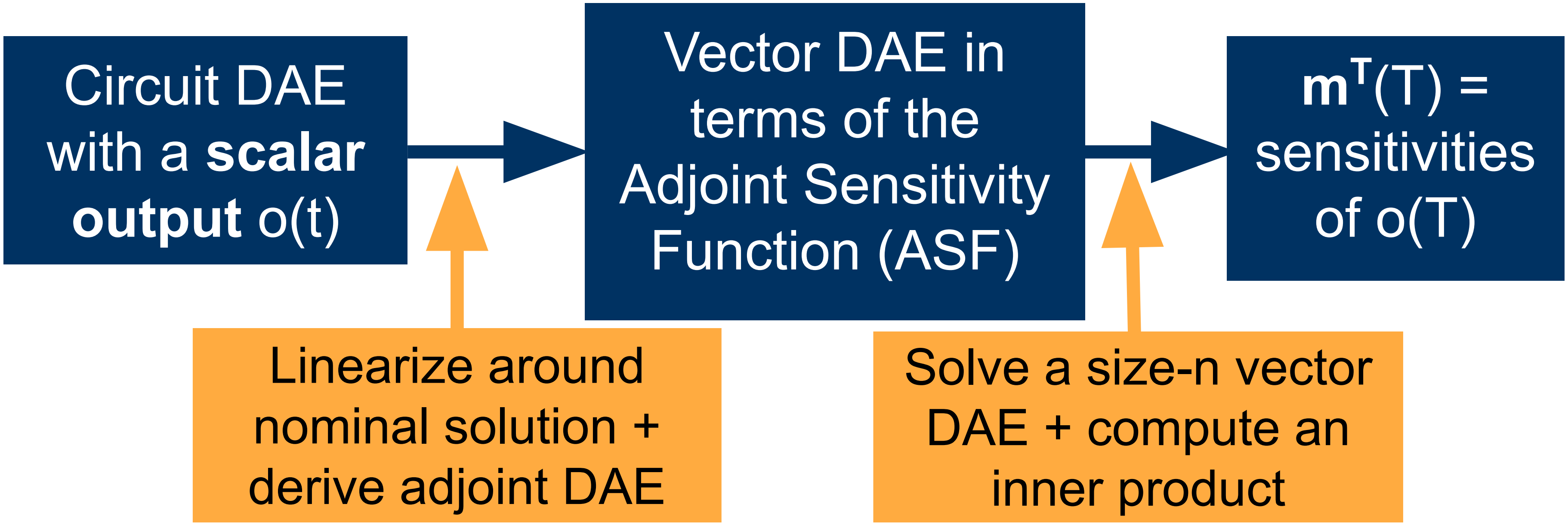


- M(t) is a dense sensitivity matrix of size $n \times n_p$, $\mathbf{M}(t) = \dfrac{\partial \vec{x}}{\partial \vec{p}}$
- Solving the sensitivity DAE has complexity $O(n n_p T)$
- This is infeasible for circuits with large numbers of parameters

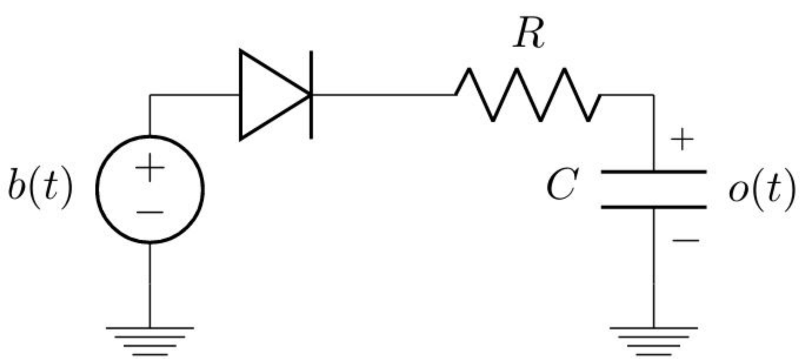


$$\mathbf{M}(t) = \begin{bmatrix} \text{Sens. of } v(t) \text{ to } R & \text{Sens. of } v(t) \text{ to } C \\ \text{Sens. of } i(t) \text{ to } R & \text{Sens. of } i(t) \text{ to } C \end{bmatrix}$$

Adjoint Sensitivity Computation



- We typically care about only a few circuit outputs
- Complexity $O((n+n_p)T)$: much more efficient than direct



$$\vec{m}^T(T) = [\text{Sens. of } o(T) \text{ to } R \quad \text{Sens. to } C]$$

Sensitivity Computation – History

- Direct computation only:
  - D. Hocevar, P. Yang, T. Trick, and B. Epler, 1985
- Specialized techniques:
  - S. Director and R. Rohrer, 1969
  - P. Feldmann, T. V. Nguyen, S. W. Director, and R. A. Rohrer, 1991

Only for a few circuit elements

simulator-specific

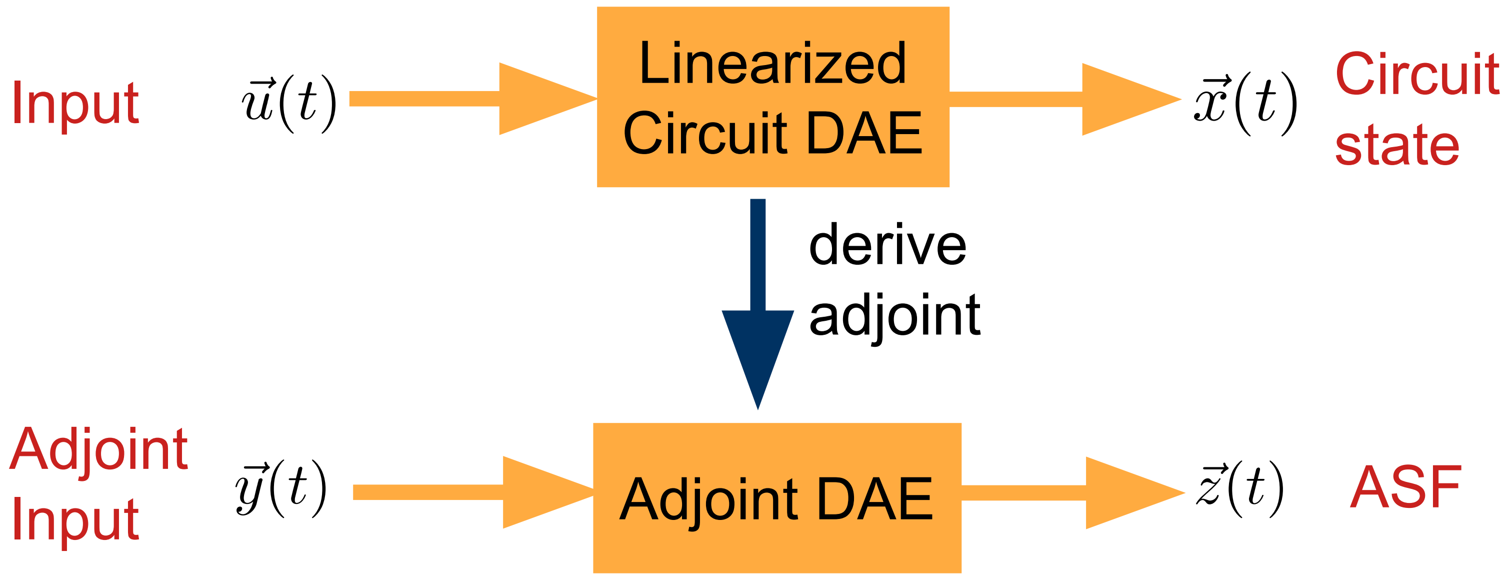
- A priori discretization:
  - F. Liu and P. Feldmann, 2014.
  - K.V. Aadithya, E. Keiter and T. Mei, 2017.

Constraints on form of DAE

- ODEs or special DAE forms:
  - Y. Cao, S. Li, L. Petzold, and R. Serban, 2003.
  - A. Meir and J. Roychowdhury, 2012.

ODE only

Adjoint Operators and Inner Products

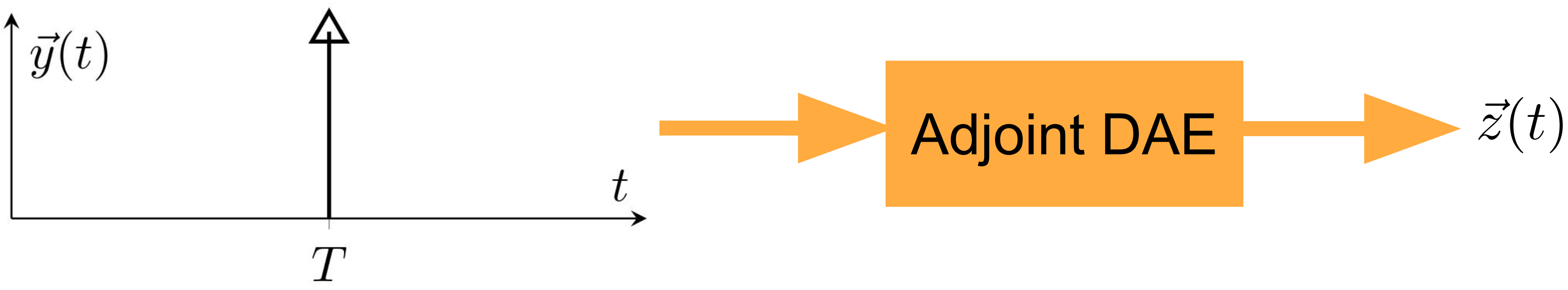


Inner product relationship: $\langle \vec{x}(t), \vec{y}(t) \rangle = \langle \vec{u}(t), \vec{z}(t) \rangle$

Note: the inner product of two vector functions is defined as

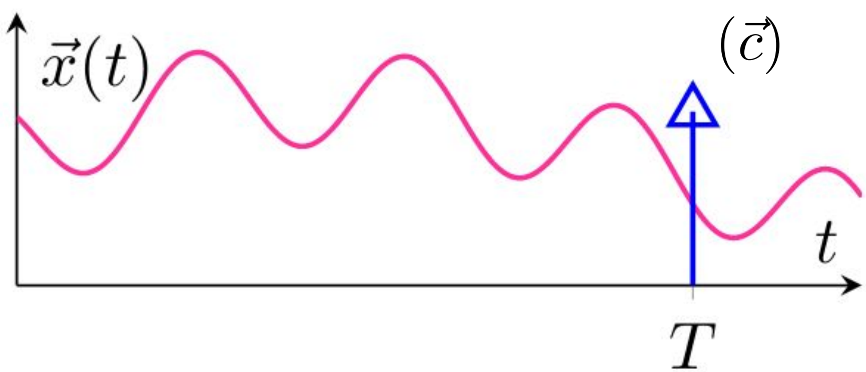
$$\langle \vec{x}(t), \vec{y}(t) \rangle = \int \vec{y}^T(t) \vec{x}(t) \, dt$$

Dirac δ Input into the Adjoint DAE



- This δ-function input allows us to isolates the sensitivities at time T via the adjoint inner product relationship.
- If we know the ASF, we can compute the sensitivities of o(T)

$$\langle \vec{x}(t), \vec{c}\delta(t-T)\rangle =$$



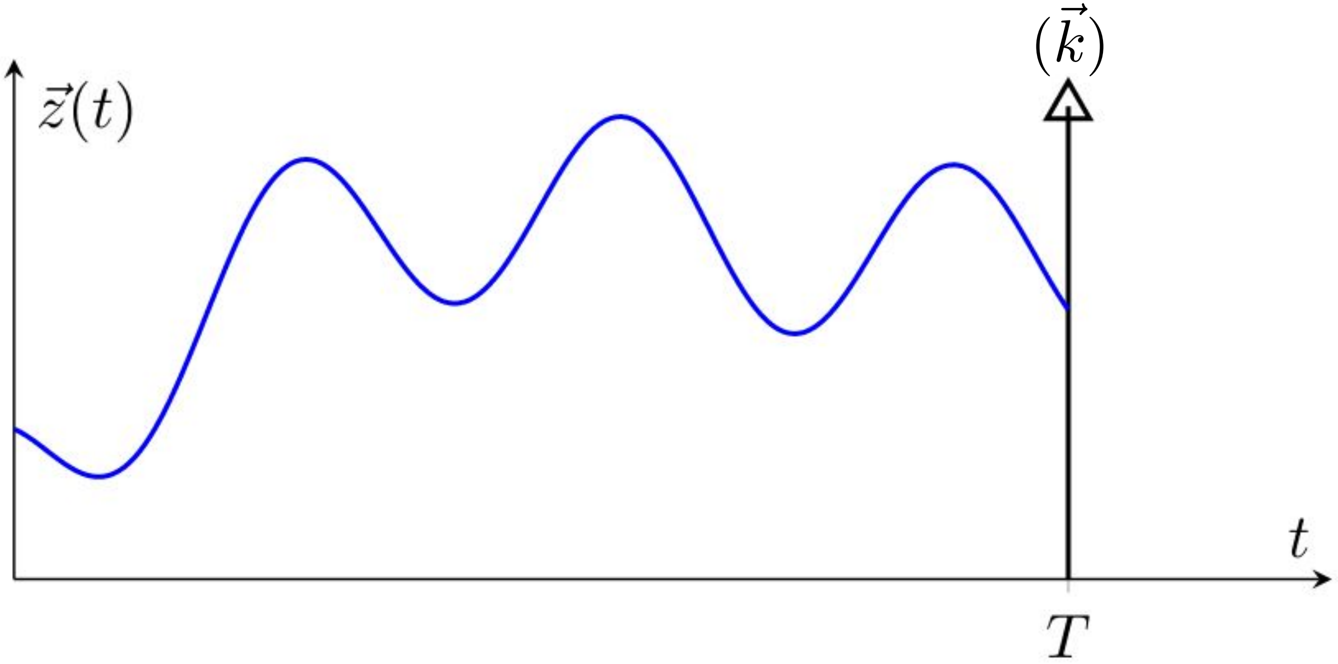
$$= \vec{c}^T \vec{x}(T)$$

Consequences of δ-Function Input

- Problem: ∞-valued, cannot be accurately represented numerically
- Solution: assume the ASF is of the form

$$\vec{z}(t) = \vec{z}_1(t) + \vec{k}\delta(t - T)$$

- and analytically derive $\vec{z}_1(t)$ and $\vec{k}$



Deriving the ASF

Infinitesimal integration of adjoint DAE at time T

↓

Derive analytical expressions for **k** and **z1**(T-)

↓

Solve the adjoint DAE backwards from T- to 0 for **z1**(t)

↓ All components in this range are finite-valued

Compute sensitivities using adjoint inner-product relationship

The Adjoint DAE

Circuit DAE

$$\frac{d}{dt}\vec{q}(\underline{\vec{x}(t)}) + \vec{f}(\underline{\vec{x}(t)}) + \vec{b}(t) = \vec{0}$$

- Solve DAE with nominal parameters for solution $\vec{x}_{\text{nom}}(t)$
- Linearize around nominal parameters and solution

Linearized DAE

$$\frac{d}{dt}\mathbf{C}(t)\underline{\Delta\vec{x}(t)} + \mathbf{G}(t)\underline{\Delta\vec{x}(t)} = \underline{-\mathbf{S}(t)\Delta\vec{p}}$$

where

$$\mathbf{C}(t) = \frac{\partial \vec{q}}{\partial \vec{x}}, \mathbf{G}(t) = \frac{\partial \vec{f}}{\partial \vec{x}}, \mathbf{S}(t) = \frac{d}{dt}\frac{\partial \vec{q}}{\partial \vec{p}} + \frac{\partial \vec{f}}{\partial \vec{p}}$$

Adjoint DAE

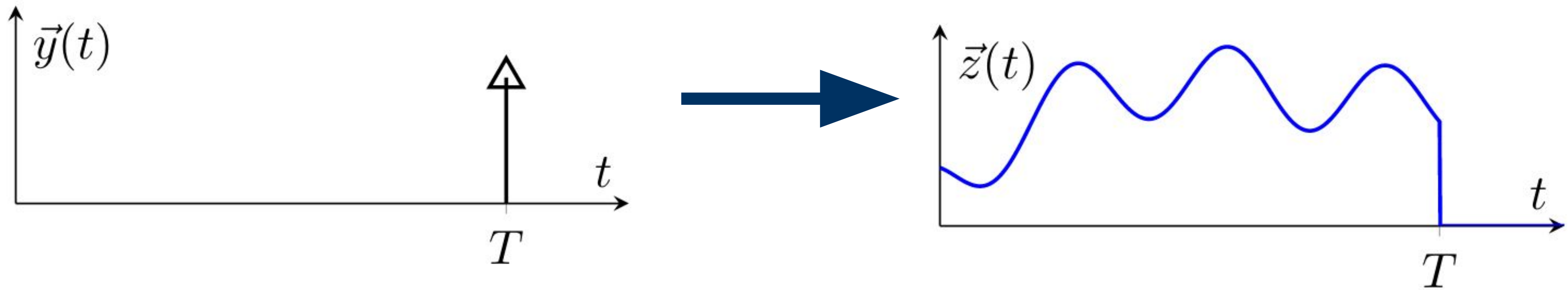
$$-\mathbf{C}^T(t)\frac{d}{dt}\vec{z}(t) + \mathbf{G}^T(t)\vec{z}(t) = \vec{y}(t)$$

ASF for ODEs and Algebraic Equations

ODEs

$$\frac{d}{dt}\vec{x}(t) + \vec{f}(\vec{x}(t)) = \vec{b}(t)$$

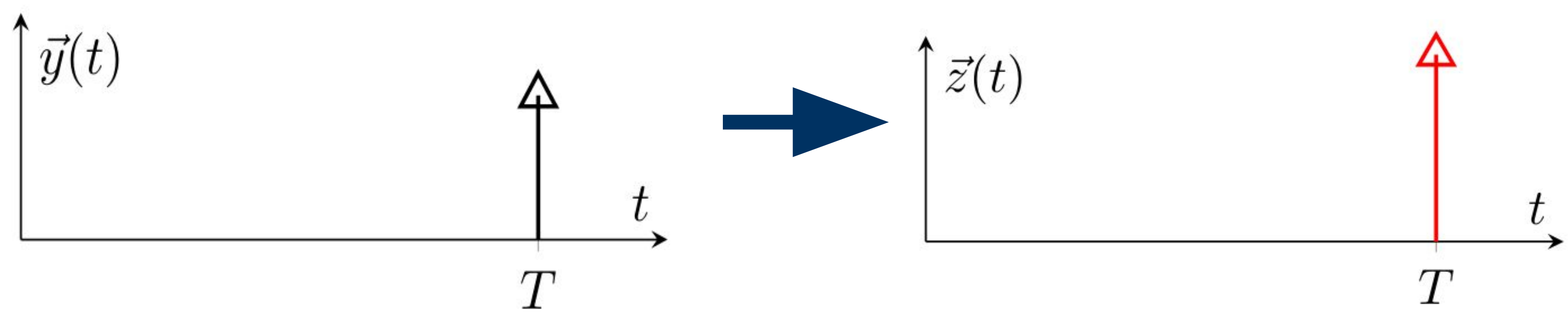
- Adjoint DAE is a linear time-varying ODE: $\frac{d}{dt}\vec{z}(t) - \mathbf{G}^T(t)\vec{z}(t) = \vec{y}(t)$



Algebraic Equations

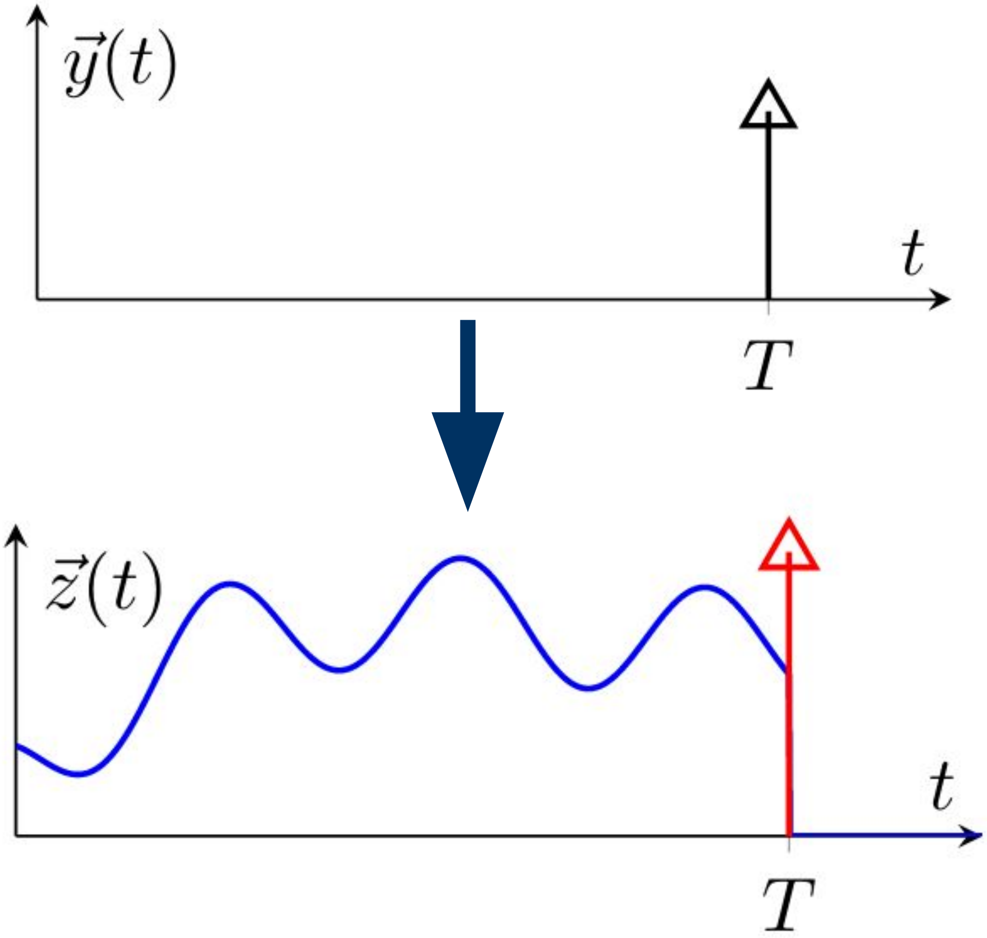
$$\vec{f}(\vec{x}(t)) = \vec{b}(t)$$

- Adjoint DAE is a system of algebraic equations: $\mathbf{G}^T(t)\vec{z}(t) = \vec{y}(t)$



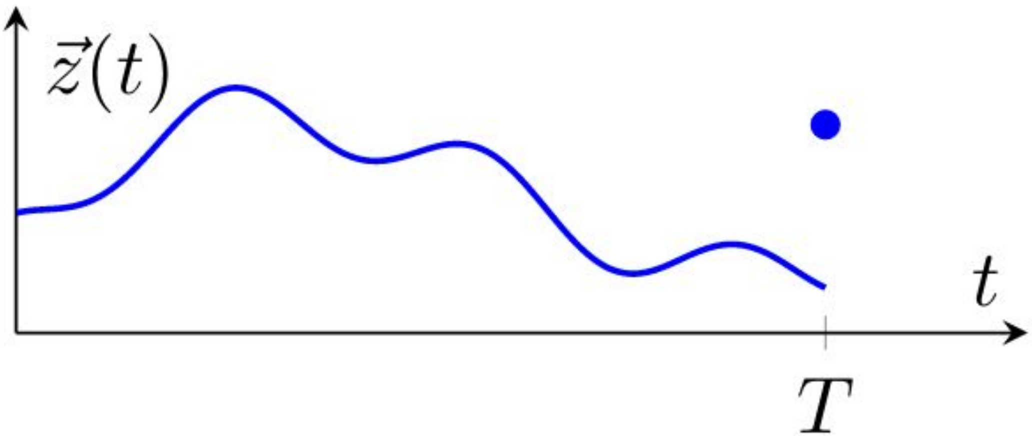
ASF for DAEs

- An impulsive input into the adjoint DAE results in an impulsive output superposed onto a finite waveform



Discontinuity in Computing the ASF

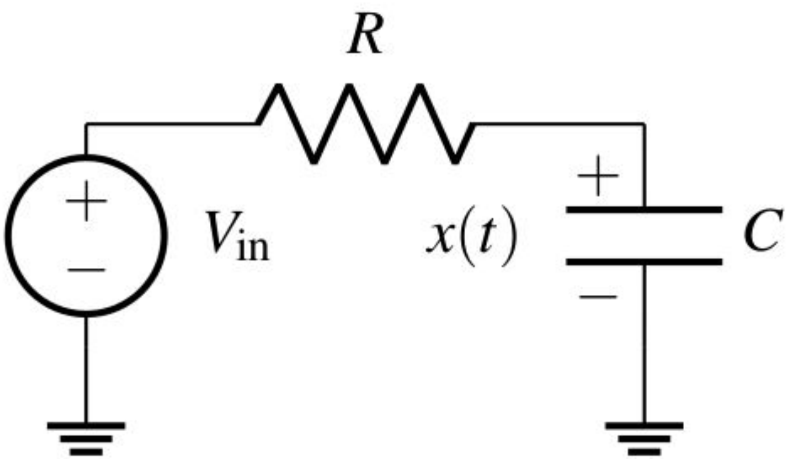
C(T) rank-deficient → Final condition **z1(T-)** may be inconsistent



- Issues in numerically integrating **z1(t)** to get final sensitivities
- **Solution**: take a small first transient timestep and discard **z1**(T)

Results: Simple Size-2 DAE

- **Goal:** Illustrate using a hand-calculable example
- **System:** size-two DAE:
- RC ODE with separate algebraic equation added on



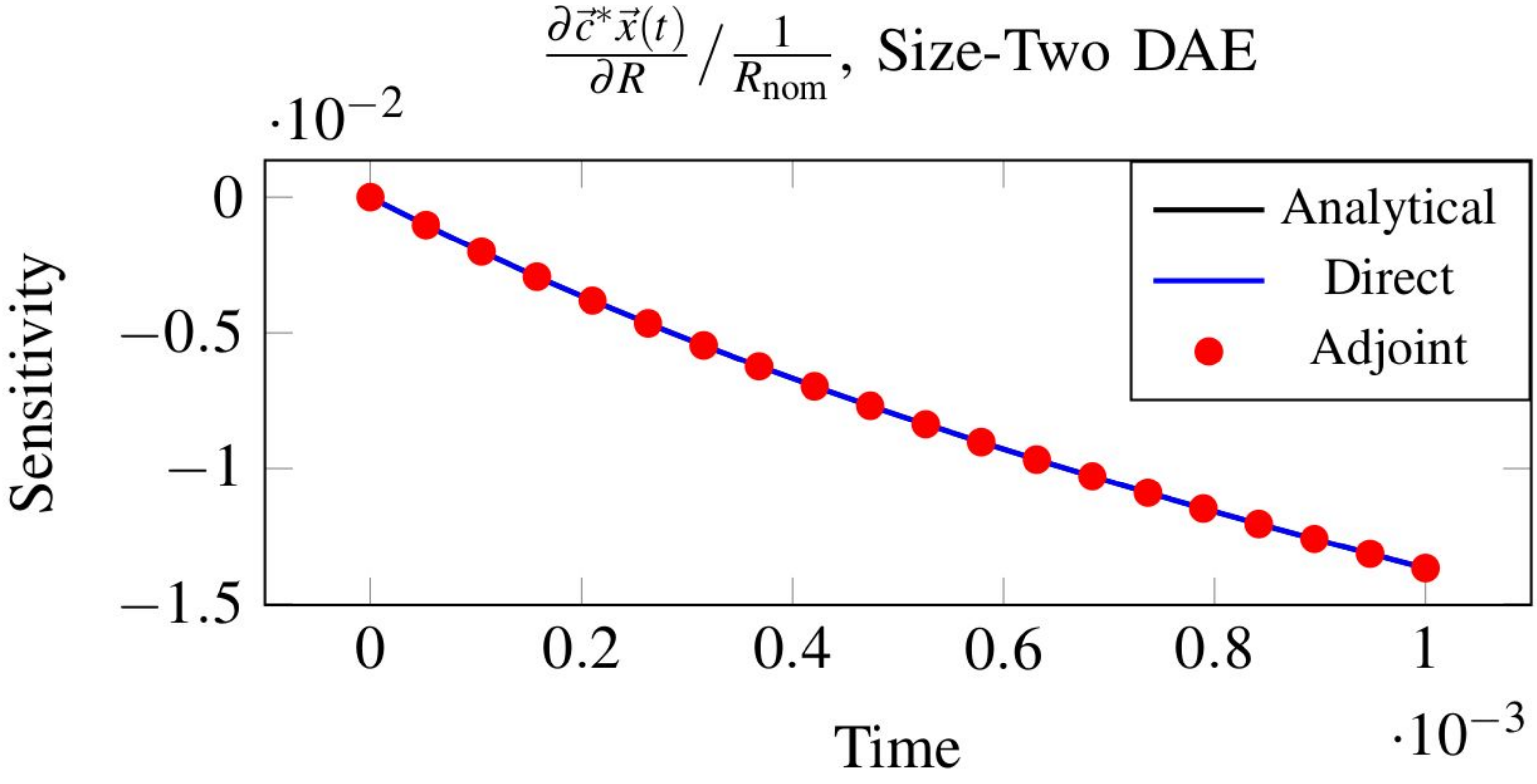
$$\begin{cases} \frac{d}{dt}x_1(t) + \frac{x_1(t) - V_{in}}{RC} = 0 \\ x_2(t) - \frac{t}{RC} = 0 \end{cases}$$

- **Analytical sensitivities:**

$$\frac{\partial \vec{x}}{\partial \vec{p}} = \begin{bmatrix} \frac{t}{R^2C}(x_1(0) - V_{in})e^{-\frac{t}{RC}} & \frac{t}{C^2R}(x_0 - V_{in})e^{-\frac{t}{RC}} \\ -\frac{t}{R^2C} & -\frac{t}{RC^2} \end{bmatrix}.$$

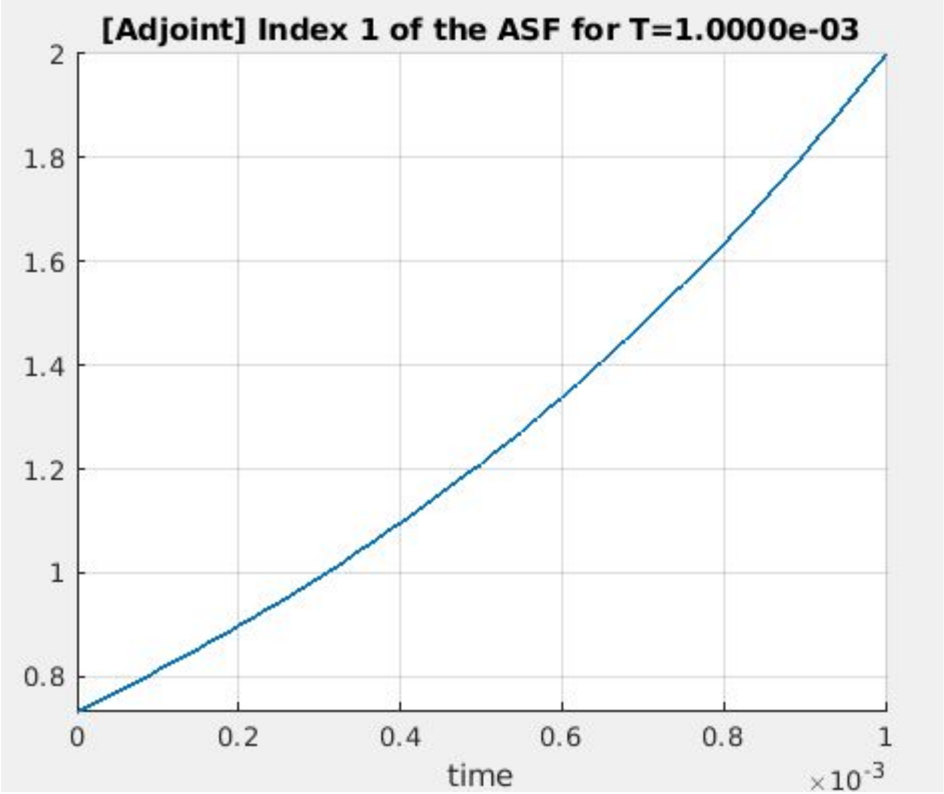
Results: Simple Size-2 DAE

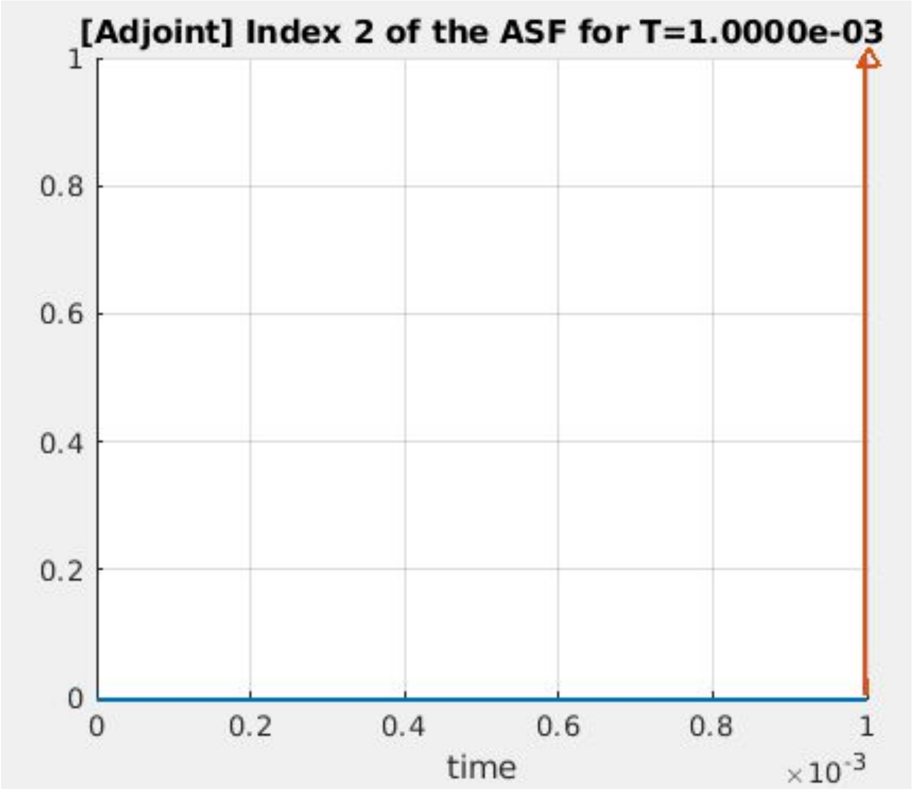
- **Setup**: output = 2**x1**(t) + **x2**(t), R = 1 kΩ, C = 1 μF, T = 1 ms
- **Plot**: Analytical, direct, and adjoint sensitivities to R over time (change in output per percent change in R)



Results: Simple Size-2 DAE

ASF for T = 1 ms, o(t) = 2x1(t) + x2(t)





Sensitivities at T:

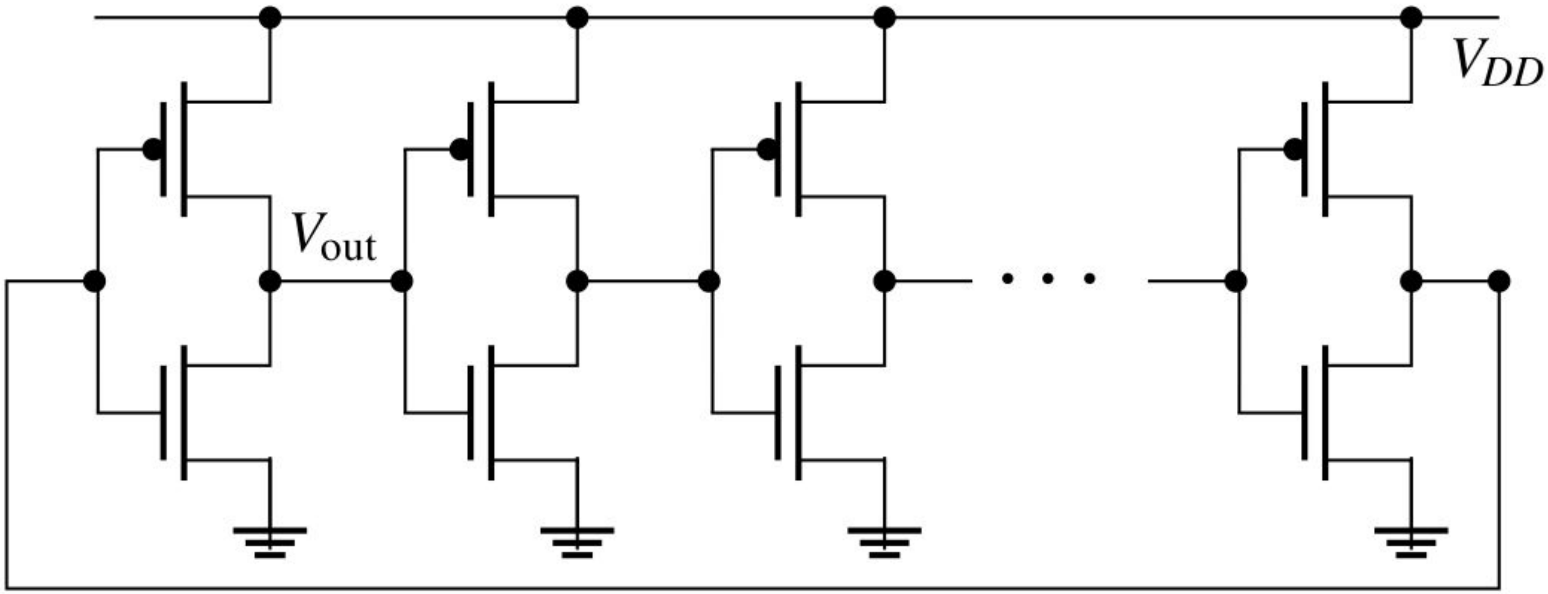
- To R: -1.3679e-03 V/Ohm
- To C: -1.3679e+06 V/F

Normalized by pNom value:

- To R: -1.3679e-02 V/%
- To C: -1.3679e-02 V/%

Results: 51-Stage MOS Ring Osc.

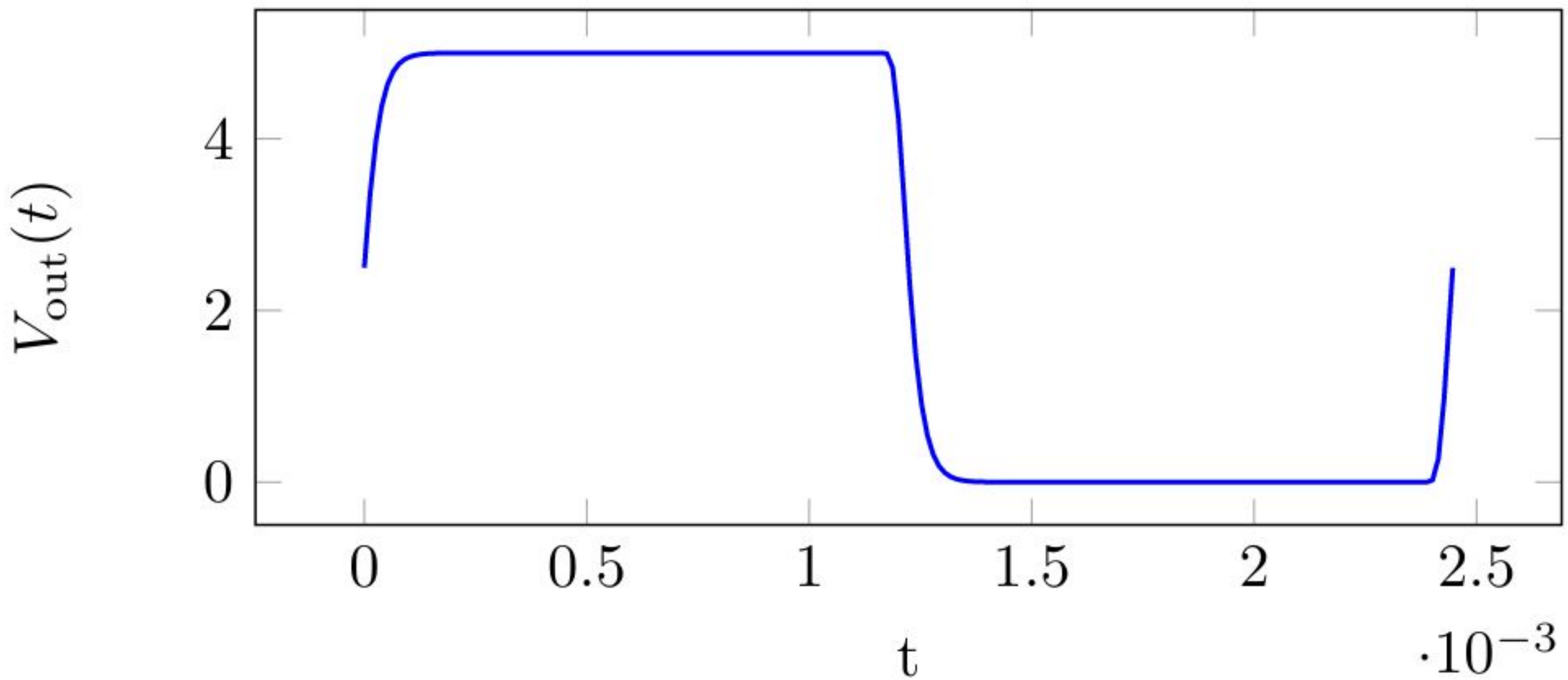
- **Goal:** Efficiency on a larger, more interesting circuit
- **System:** 51-stage MOS ring oscillator



- **155 unknowns, 664 parameters**

Results: 51-Stage MOS Ring Osc.

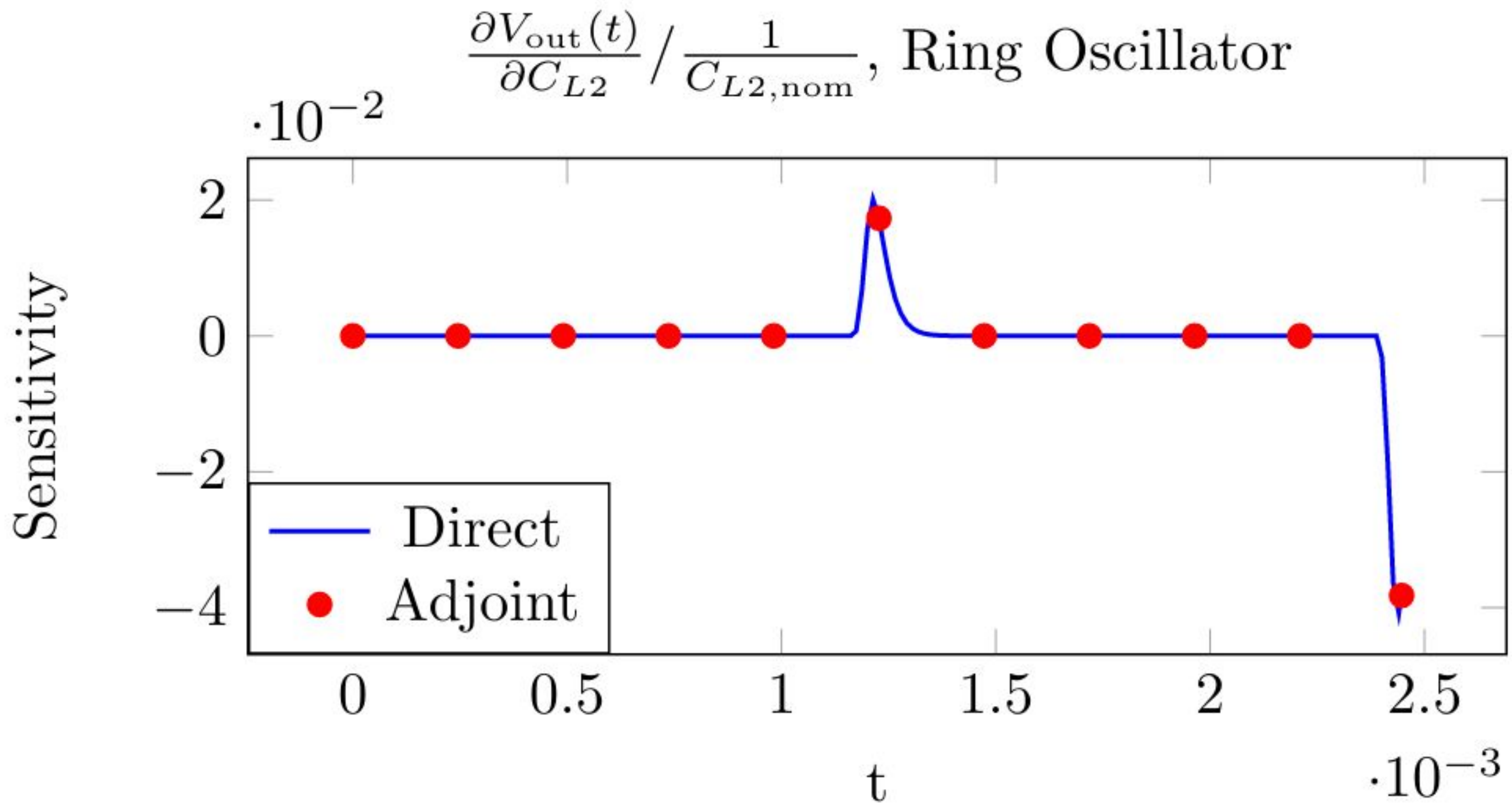
Nominal Output Waveform



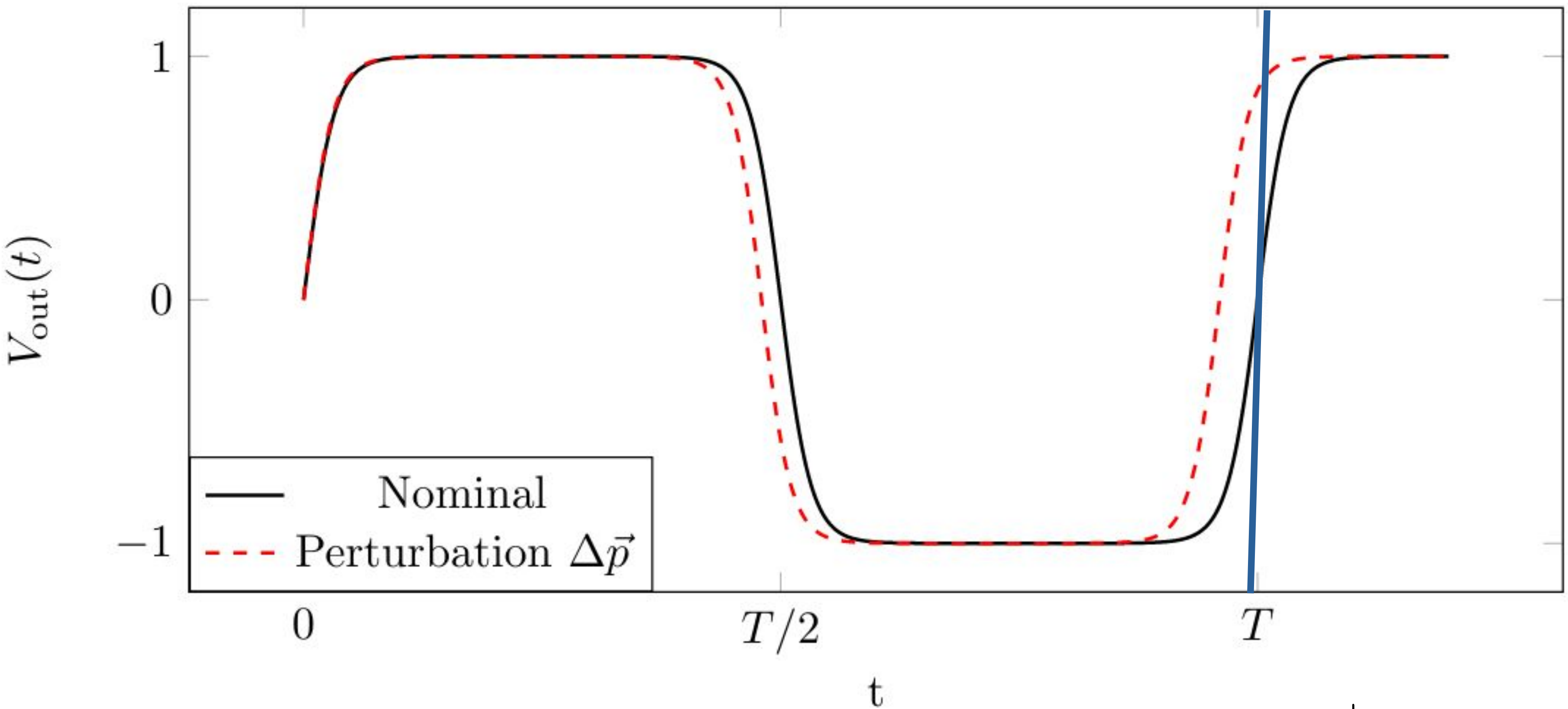
- **Setup**: 3000 timepoints over one oscillator period (T = 2.447 ms)
- **Most important parameters**: load capacitors at output of each inverter
- **Runtime: direct = 7392 s, adjoint = 24.6 s**

Results: 51-Stage MOS Ring Osc.

- **Plot:** sensitivities of output to load cap at output
- Normalized by nominal parameter value



Results: Sensitivity of Ring Oscillator Period



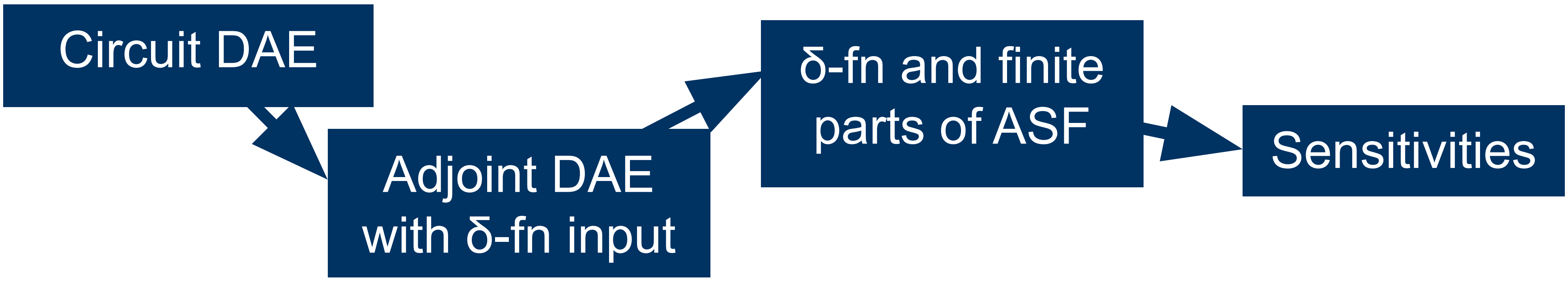
- **Want**: ΔT, given ΔV/Δ**p** and dV/dt: $$\frac{\partial T_{\text{osc}}}{\partial \vec{p}} = \frac{\partial v(t)/\partial \vec{p}\big|_{T_{\text{nom}}}}{dv(t)/dt\big|_{T_{\text{nom}}}}$$
- **Result**: a 1% change in the output load capacitance will cause a 0.0209% change in the oscillator period

Summary

Problem

- Devise an efficient and numerically well-defined method for computing sensitivities of circuit outputs via adjoints

Process of Deriving Sensitivities



Results

- **Accurate + faster than direct**